# Organizing Physics with Open Energy-Driven Systems

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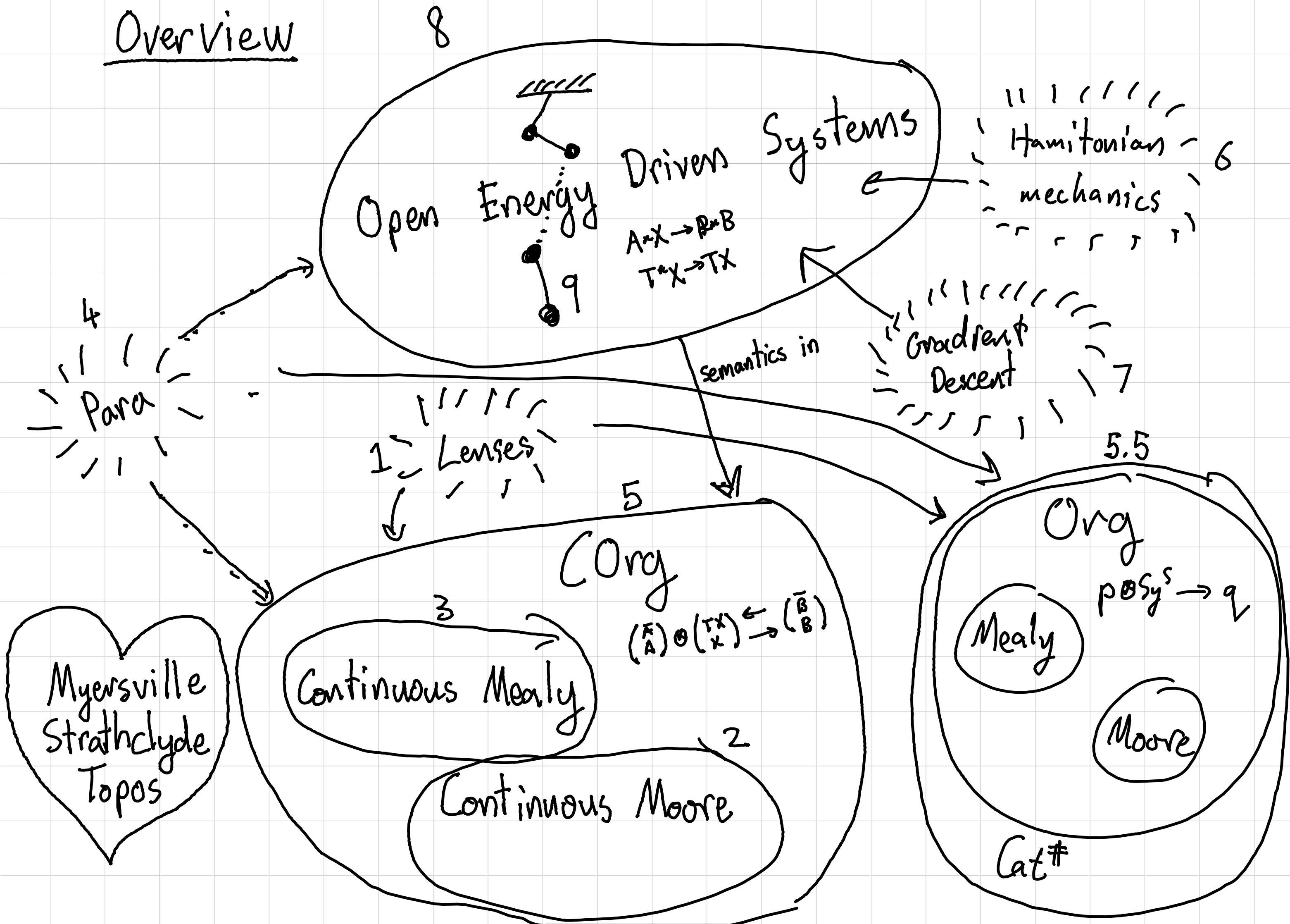
# Overview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Open Energy Driven Systems
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COrg
$\binom{\bar{A}}{A} \otimes \binom{TX}{X} \leftrightarrows \binom{\bar{B}}{B}$
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Continuous Mealy
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Continuous Moore
5.5
Org
$p \otimes \mathcal{y}^{\mathcal{S}} \to q$
Mealy
Moore
Cat#
Myersville Strathclyde Topos

# Part 1: Building machinery for the "semantics"[1] of our energy-driven systems.

$$\begin{pmatrix} \bar{A} \\ A \end{pmatrix} \otimes \begin{pmatrix} TX \\ X \end{pmatrix} \rightleftarrows \begin{pmatrix} \bar{B} \\ B \end{pmatrix}$$

"Lenses and Para"

1 Where semantics just means "the codomain of a functor."

# 1 Lenses

$$\begin{pmatrix} \bar{A} \\ A \end{pmatrix} \underset{f}{\overset{f^\#}{\leftrightarrows}} \begin{pmatrix} \bar{B} \\ B \end{pmatrix}$$

**Simple:** $\underset{\text{set}}{A} \overset{f}{\dashrightarrow} \underset{\text{set}}{B}$ , $A \times \underset{\text{set}}{\bar{B}} \overset{f^\#}{\dashrightarrow} \underset{\text{set}}{\bar{A}}$

**Dependent:**

$$\begin{tikzcd}
\bar{A} \arrow[d, "\pi_A"'] & \cdot \arrow[l, dashed, "f^\#"'] \arrow[r] \arrow[dl] \arrow[dr, phantom, "\lrcorner" very near start] & \bar{B} \arrow[d, "\pi_B"] \\
A \arrow[rr, dashed, "f"'] & & B
\end{tikzcd}$$

$f^\# : f^*(\bar{B}) \longrightarrow \bar{A}$

$\mathrm{Set}/_A \simeq \mathrm{Set}^A$

$\boxed{\text{or}}$

$$\begin{tikzcd}
p(1) \arrow[rr, "f"] \arrow[dr, "{p[-]}"'] & \arrow[d, phantom, "\overset{f^\#}{\Leftarrow}"] & q(1) \arrow[dl, "{q[-]}"] \\
& \mathrm{Set} &
\end{tikzcd}$$

$f^\# : q[f(i)] \longrightarrow p[i]$

## "Smooth" Lenses (MfdLens)

$$
\begin{pmatrix} \bar{A} \\ A \end{pmatrix} \underset{f}{\overset{f^\#}{\leftrightarrows}} \begin{pmatrix} \bar{B} \\ B \end{pmatrix}
$$

$$
\begin{tikzcd}
\bar{A} \arrow[d, two heads, "\pi_A"'] & {} \arrow[l, dotted, "f^\#"'] \arrow[r] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & \bar{B} \arrow[d, two heads, "\pi_B"] \\
A \arrow[rr, dotted, "f"'] & & B
\end{tikzcd}
$$

submersion

manifold

Submersion: $s : X \longrightarrow Y$ s.t. $s'(T_x X) = T_{s(x)} Y$

"surjective on tangent spaces $\Rightarrow$ plays nice with pullback"

## Example: Cotangent

$$T^* : \mathrm{Mfd} \longrightarrow \mathrm{MfdLens}$$

$$X \xrightarrow{f} Y \quad \longmapsto \quad \begin{pmatrix} T^*X \\ X \end{pmatrix} \underset{f}{\overset{df}{\leftrightarrows}} \begin{pmatrix} T^*Y \\ Y \end{pmatrix}$$

$$X \xrightarrow{f} Y \quad \longmapsto \quad \begin{pmatrix} \mathbb{R}^X \\ X \end{pmatrix} \underset{f}{\overset{f ; -}{\leftrightarrows}} \begin{pmatrix} \mathbb{R}^Y \\ Y \end{pmatrix}$$

# Lenses in general (Spivak 2019!)

$$\mathrm{Bun} : \mathrm{Spc}^{op} \longrightarrow \mathrm{Cat}$$

(Spc = Set, Mfd, etc., Bun = $A \mapsto$ Simple $\begin{tikzcd} A \times B \arrow[r] \arrow[dr, "\pi_1"'] & A \times B' \arrow[d, "\pi_1"] \\ & A \end{tikzcd}$ , $A \mapsto$ Dependent $\begin{tikzcd} \bar{A} \arrow[r] \arrow[dr] & \bar{A}' \arrow[d] \\ & A \end{tikzcd}$ , $X \mapsto$ $\begin{tikzcd} \bar{X} \arrow[r] \arrow[dr] & \bar{X}' \arrow[d] \\ & X \end{tikzcd}$ Smooth, etc.)

$$\mathrm{Lens}(\mathrm{Bun}) := \int_{X : \mathrm{Spc}} \mathrm{Bun}(X)^{op}$$

$$\downarrow$$

$$\mathrm{Spc}$$

$$\begin{pmatrix} \bar{A} \\ A \end{pmatrix} : (A : \mathrm{Spc}, \bar{A} : \mathrm{Bun}(A))$$

$$\begin{tikzcd} \begin{pmatrix} \bar{A} \\ A \end{pmatrix} \arrow[r, shift right, dashed, "f"'] & \begin{pmatrix} \bar{B} \\ B \end{pmatrix} \arrow[l, shift right, dashed, "f^\#"'] \end{tikzcd} \qquad f : A \longrightarrow B \quad f^\# : f^*(\bar{B}) \longrightarrow \bar{A}$$

# 2 Continuous Moore Machines

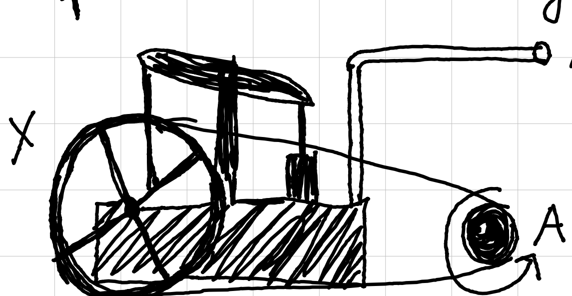
$$\begin{pmatrix} TX \\ X \end{pmatrix} \underset{f}{\overset{f^\#}{\leftrightarrows}} \begin{pmatrix} \bar{A} \\ A \end{pmatrix}$$

$f =$ "observation"
$f^\# =$ "update"

$a = f(x)$
$\dot{x} = f^\#(x, \bar{a})$

$\bar{a}$ "input" depends on $a$ "output"

Example: Controllable system




input: steam pressure and flow rate
output: rotational velocity

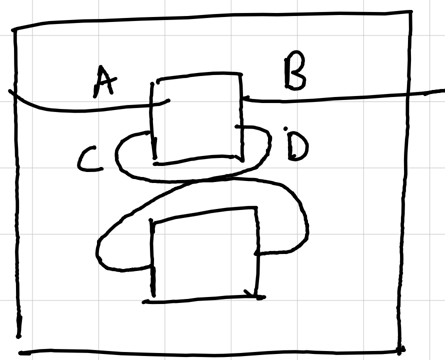
# Operad Algebra of Moore Machines

$$\begin{pmatrix}\bar{A}\\ A\end{pmatrix} \longmapsto \begin{tikzcd}[column sep=small] \begin{pmatrix}TX\\ X\end{pmatrix} \arrow[rr, shift left, "f'"] \arrow[rr, shift right, "f"'] \arrow[dr, shift left] & & \begin{pmatrix}TX'\\ X'\end{pmatrix} \arrow[dl, shift left] \\ & \begin{pmatrix}\bar{A}\\ A\end{pmatrix} \arrow[ul, shift left] \arrow[ur, shift left] & \end{tikzcd}$$

$$\mathrm{Sys} : \mathrm{MfdLens} \to \mathrm{Cat}$$

$$\begin{pmatrix}\bar{A}\\ A\end{pmatrix} \otimes \begin{pmatrix}\bar{B}\\ B\end{pmatrix} = \begin{pmatrix}\bar{A}\times\bar{B}\\ A\times B\end{pmatrix}$$

$$\mathrm{Sys}\left(\begin{pmatrix}\bar{A}\\ A\end{pmatrix}\right) \times \mathrm{Sys}\left(\begin{pmatrix}\bar{B}\\ B\end{pmatrix}\right) \longrightarrow \mathrm{Sys}\left(\begin{pmatrix}\bar{A}\\ A\end{pmatrix} \otimes \begin{pmatrix}\bar{B}\\ B\end{pmatrix}\right)$$




$$\begin{tikzcd} \begin{pmatrix}A\times C\times B\times D\\ B\times D\end{pmatrix} \arrow[d, shift left] & \otimes & \begin{pmatrix}D\times C\\ C\end{pmatrix} \arrow[d, shift left] \arrow[r, shift right] & \begin{pmatrix}A\times B\\ B\end{pmatrix} \arrow[l, shift right] \\ \begin{pmatrix}TX\\ X\end{pmatrix} \arrow[u, shift left] & & \begin{pmatrix}TY\\ Y\end{pmatrix} \arrow[u, shift left] & \end{tikzcd}$$

# 3 Continuous Mealy Machine (thanks Eigil!)

$$\begin{pmatrix}\bar{A}\\A\end{pmatrix}\otimes\begin{pmatrix}TX\\X\end{pmatrix} \underset{f}{\overset{f^\#}{\leftrightarrows}} \begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}\bar{A}\\A\end{pmatrix}\otimes\begin{pmatrix}TX\\X\end{pmatrix}$$

"a stateful section of $\begin{pmatrix}\bar{A}\\A\end{pmatrix}$"

$f =$ nothing

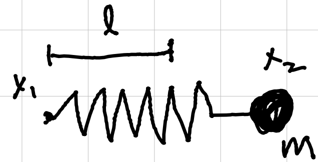
$f^\#_1 =$ "update"

$f^\#_2 =$ "feedback"

$$\dot{x} = f^\#_1(x,a)$$
$$\bar{a} = f^\#_2(x,a)$$

$\bar{a}$ "output" depends on $a$ "input"

## Example: spring-and-mass




$$\begin{pmatrix} T^*T\mathbb{R} \\ T\mathbb{R} \end{pmatrix} \otimes \begin{pmatrix} TT^*\mathbb{R} \\ T^*\mathbb{R} \end{pmatrix} \rightleftarrows \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

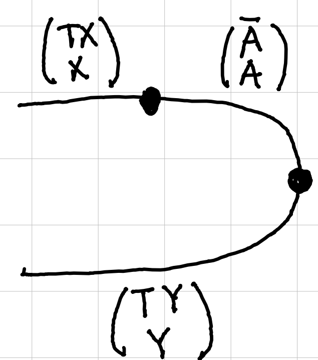
input $= (x_1, v_1)$, state $= (x_2, p_2)$ feedback = force of spring

$$\dot{x}_2 = \frac{1}{m} p_2 .$$
$$\dot{p}_2 = -k((x_2 - x_1) - l)$$

$$dx_1 = k((x_2 - x_1) - l)$$
$$dv_1 = 0$$

# Composing Moore and Mealy

$$\begin{pmatrix} TX \\ X \end{pmatrix} \leftrightarrows \begin{pmatrix} \bar{A} \\ A \end{pmatrix} \qquad \begin{pmatrix} \bar{A} \\ A \end{pmatrix} \otimes \begin{pmatrix} TY \\ Y \end{pmatrix} \leftrightarrows \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$




Goal $= \begin{pmatrix}\bar{A}\\ A\end{pmatrix} \otimes \begin{pmatrix}TX\\ X\end{pmatrix} \leftrightarrows \begin{pmatrix}\bar{B}\\ B\end{pmatrix}$

When $\begin{pmatrix}\bar{A}\\ A\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}$: Moore,
$\begin{pmatrix}\bar{B}\\ B\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}$: Mealy

$$\begin{tikzcd} \begin{pmatrix}\bar{A}\\ A\end{pmatrix} \arrow[r, "X", "\shortmid" marking] & \begin{pmatrix}\bar{B}\\ B\end{pmatrix} \arrow[r, "Y", "\shortmid" marking] \arrow[d] & \begin{pmatrix}\bar{C}\\ C\end{pmatrix} \\ & \begin{pmatrix}\bar{A}\\ A\end{pmatrix} \xrightarrow{X \times Y} \begin{pmatrix}\bar{C}\\ C\end{pmatrix} & \end{tikzcd}$$

"Mealy-Moore Machines"

## 4. Para construction

"State SMC" $(\mathbb{A}, \otimes_{\mathbb{A}}, I_{\mathbb{A}})$

"Interface SMC" $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$

"Action"

$$\mathcal{C} \times \mathbb{A} \xrightarrow{\bullet} \mathcal{C}$$

$\mathrm{Para}(\mathbb{A}, \mathcal{C}, \bullet)$ is symmetric monoidal double cat

- objects + vert morphisms from $\mathcal{C}$
- horizontal morphisms from $A$ to $B$ are

$$X \in \mathbb{A} \qquad A \bullet X \to B$$

# Para continued

2-cell

$$\begin{tikzcd} A_1 \arrow[r, "{(X_1,m_1)}", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B_1 \arrow[d, "g"] \\ A_2 \arrow[r, "{(X_2,m_2)}"', "|" marking] & B_2 \end{tikzcd}$$

is

$\alpha : X_1 \longrightarrow X_2$ such that

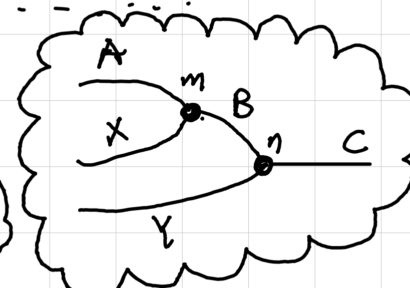
$$\begin{tikzcd} A_1 \bullet X_1 \arrow[r, "m_1"] \arrow[d, "f \bullet \alpha"'] & B_1 \arrow[d] \\ A_2 \bullet X_2 \arrow[r, "m_2"'] & B_2 \end{tikzcd}$$

Composition:

$$A \xrightarrow{(X,m)} B \xrightarrow{(Y,n)} C$$

$$\mapsto$$

$$A \bullet (X \times Y) \xrightarrow{\sim} (A \bullet X) \bullet Y \xrightarrow{m \bullet 1_Y} B \bullet Y \xrightarrow{n} C$$



## 5 <u>COrg</u>

"State SMC" $\quad\quad\quad\quad\quad\quad$ "Interface SMC"

$(\mathrm{Mfd}_{\cong}, \times, 1)$ $\quad\quad\quad\quad\quad\quad$ $\left(\mathrm{MfdLens}, \otimes, \binom{1}{1}\right)$

$T : \mathrm{Mfd}_{\cong} \longrightarrow \mathrm{MfdLens}$

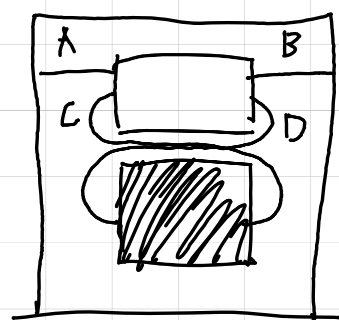
$$\binom{\bar{A}}{A} \bullet X := \binom{\bar{A}}{A} \otimes \binom{TX}{X}$$

= Para magic

COrg: a symmetric monoidal double cat with h-arrows

$$\binom{\bar{A}}{A} \otimes \binom{TX}{X} \leftrightarrows \binom{\bar{B}}{B}$$ "Mealy-Moore Machines"

# Half-filled wiring diagrams are Mealy-Moore

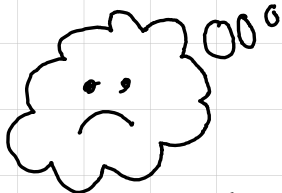



$$\begin{pmatrix} A \times C \times B \times D \\ B \times D \end{pmatrix} \otimes \begin{pmatrix} TY \\ Y \end{pmatrix} \leftrightarrows \begin{pmatrix} A \times B \\ B \end{pmatrix}$$

Bonus: Org (sort of)

"State SMC"
$(\mathrm{Set}_{\cong}, \times, 1)$

"Interface SMC"
$(\mathrm{Poly}, \otimes, y)$



Para magic

$$p \cdot S := p \otimes S y^S$$

Org

## Sidebar: what should morphisms between systems be?

Recall: A <u>chart</u> is a map

$$\begin{pmatrix}\bar{A}\\ A\end{pmatrix} \overset{f^\flat}{\underset{f}{\dashrightarrow}} \begin{pmatrix}\bar{B}\\ B\end{pmatrix} \qquad \begin{array}{l} f: A \to B \\ f^\flat: \bar{A} \to f^*(\bar{B}) \end{array}$$

(Grothendieck construction of Bun)

$$T: \mathrm{Mfd} \to \mathrm{MfdChart}$$

2-cell:

$$\begin{tikzcd}
\begin{pmatrix}\bar{A}_1\\ A_1\end{pmatrix} \otimes \begin{pmatrix}TX_1\\ X_1\end{pmatrix} \arrow[d, shift right] \arrow[d, shift left] \arrow[r, shift left] & \begin{pmatrix}\bar{B}_1\\ B\end{pmatrix} \arrow[l, shift left] \arrow[d, shift right] \arrow[d, shift left] \\
\begin{pmatrix}\bar{A}_2\\ A_2\end{pmatrix} \otimes \begin{pmatrix}TX_2\\ X_2\end{pmatrix} \arrow[r, shift left] & \begin{pmatrix}\bar{B}_2\\ B_2\end{pmatrix} \arrow[l, shift left]
\end{tikzcd}$$

$f \quad Tf$

# Problem: What is the clock system?

For Moore machines, one "clock system" is

$$\begin{tikzcd}
\begin{pmatrix} T\mathbb{R} \\ \mathbb{R} \end{pmatrix} \arrow[r, shift right] \arrow[d, shift right] \arrow[d, shift left] & \begin{pmatrix} T\mathbb{R} \\ \mathbb{R} \end{pmatrix} \arrow[l, shift right] \arrow[d, shift right] \arrow[d, shift left] \\
\begin{pmatrix} TX \\ X \end{pmatrix} \arrow[r, shift right] & \begin{pmatrix} \hat{A} \\ A \end{pmatrix} \arrow[l, shift right]
\end{tikzcd}$$

But.... $\begin{pmatrix} T\mathbb{R} \\ \mathbb{R} \end{pmatrix} \otimes \begin{pmatrix} T\mathbb{R} \\ \mathbb{R} \end{pmatrix} \leftrightarrows \begin{pmatrix} T\mathbb{R} \\ \mathbb{R} \end{pmatrix}$ doesn't make sense...

A fun puzzle for the audience,
not necessary for rest of talk

Ask a question!

Chat with your neighbor quietly!

# Intermission

(5 min)

Stretch!

Eat a crunchy snack!

# Part 2: Mealy-Moore machines that are derived from potentials

$$X \xrightarrow{P} \mathbb{R} \quad \longmapsto \quad \begin{tikzcd} T^*X \arrow[d, bend right, "p"'] \\ X \arrow[u, bend right, "dP"'] \end{tikzcd}$$

if:
$$\begin{tikzcd} TX \arrow[dr] & & T^*X \arrow[ll, "R"'] \arrow[dl] \\ & X & \end{tikzcd}$$

then:
$$\begin{tikzcd} TX \arrow[dr, shift left] & & T^*X \arrow[ll, "R"'] \arrow[dl, shift right] \\ & X \arrow[ul, shift left, "R \circ dP"] \arrow[ur, shift right, "dP"'] & \end{tikzcd}$$

Do this... but open-systems style!

# 6. Reaction structures for gradient descent

If $X$ <u>Riemannian</u> then we have

$$\begin{tikzcd} TX \arrow[dr] & & T^*X \arrow[ll, "M"'] \arrow[dl] \\ & X & \end{tikzcd}$$

"Reaction structure"

coming from metric tensor

<u>Example:</u> $\mathbb{R}^n$

Use this on $\underset{\text{"score"}}{S} : X \to \mathbb{R}$ to get $\dot{x} = \nabla S$

$M(x) \cdot dS(x)$

$$\frac{dS}{dt} = \langle \dot{x}, \nabla S \rangle = \|\nabla S\|^2 \geq 0$$

# 7. Reaction structures for Hamiltonian mechanics

If $X$ is Poisson (stronger = symplectic, weaker = almost Poisson)

"Reaction structure"

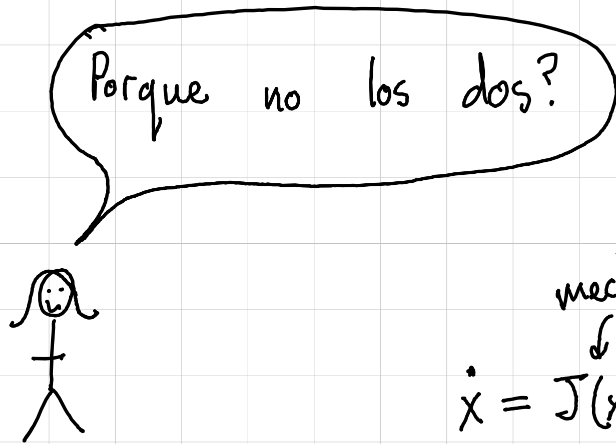
$$\begin{tikzcd} TX \arrow[dr] & & T^*X \arrow[ll, "J"'] \arrow[dl] \\ & X & \end{tikzcd}$$

(derived from Poisson bracket)

<u>Example</u>: $T^*Q = X$, $H$ is "Hamiltonian" energy fn.

$$\dot{x} = J(x)\, dH(x) \iff \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix}$$

$$\frac{dH}{dt} = \langle \dot{x}, dH(x) \rangle = \langle J(x)\, dH(x), dH(x) \rangle = 0$$




# Non-equilibrium thermo

mechanics (antisymmetric) Energy friction (non-negative definite) entropy

$$\dot{x} = \overset{\text{mechanics}}{J(x)}\, \overset{\text{Energy}}{dH(x)} + \overset{\text{friction}}{M(x)}\, \overset{\text{entropy}}{dS(x)}$$

$$J(x)\, dS(x) = 0 \quad , \quad M(x)\, dH(x) = 0$$

"GENERIC equation"

$$\dot{x} = \underbrace{\big(J(x) - \beta M(x)\big)}_{\text{Reaction}} \underbrace{\big(dH(x) - \tfrac{1}{\beta} dS(x)\big)}_{\text{Exergy}}$$

# 8. Open Energy-driven systems

Suppose ...

- $X : \mathrm{Mfd}$, with
$$\begin{tikzcd} TX \arrow[dr] & T^*X \arrow[l, "R"'] \arrow[d] \\ & X \end{tikzcd}$$
a reaction
- $A \times X \xrightarrow{f} B \times \mathbb{R}$ is a smooth function.

Then ...

$$\begin{pmatrix} T^*A \\ A \end{pmatrix} \otimes \begin{pmatrix} TX \\ X \end{pmatrix} \overset{R}{\underset{=}{\leftrightarrows}} \begin{pmatrix} T^*A \\ A \end{pmatrix} \otimes \begin{pmatrix} T^*X \\ X \end{pmatrix} \overset{df}{\underset{f}{\leftrightarrows}} \begin{pmatrix} T^*B \\ B \end{pmatrix} \otimes \begin{pmatrix} T^*\mathbb{R} \\ \mathbb{R} \end{pmatrix} \overset{\langle id, dx \rangle}{\underset{\pi_1}{\leftrightarrows}} \begin{pmatrix} T^*B \\ B \end{pmatrix}$$

is a Mealy-Moore Machine

# The SMC of Open Energy-Driven Systems

"State SMC"

$\mathrm{React} : \mathrm{Mfd}_{\cong} \longrightarrow \mathrm{Set}$

$$X \longmapsto \left\{ \begin{array}{c} TX \leftarrow T^*X \\ \searrow \ \swarrow \\ X \end{array} \right\}$$

React is lax monoidal

$\int \mathrm{React}$ is SMC

"Interface SMC"

$(\mathrm{Mfd}, \times, 1)$

"Action"

$A \bullet (X, R) = A \times X$

I want to be a double category...

"Para magic"

+ (local connected components)

$\Downarrow$

OpenReact $\xrightarrow{\mathrm{Kl}_{- \times R}(-)}$ OpenErg

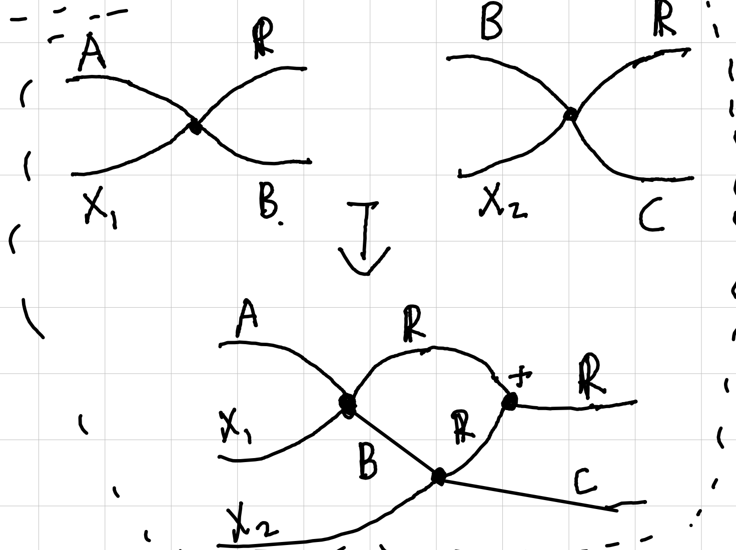
# Composition in Open Erg

$$\begin{tikzcd}
A \arrow[r, "{(X_1, R_1, H_1)}", "|" marking] & B \arrow[r, "{(X_2, R_2, H_2)}", "|" marking] & C \\
& \mapsto & \\
A \arrow[rr, "{(X_1 \times X_2, R_1 \times R_2, H_1 + H_2)}", "|" marking] & & C
\end{tikzcd}$$

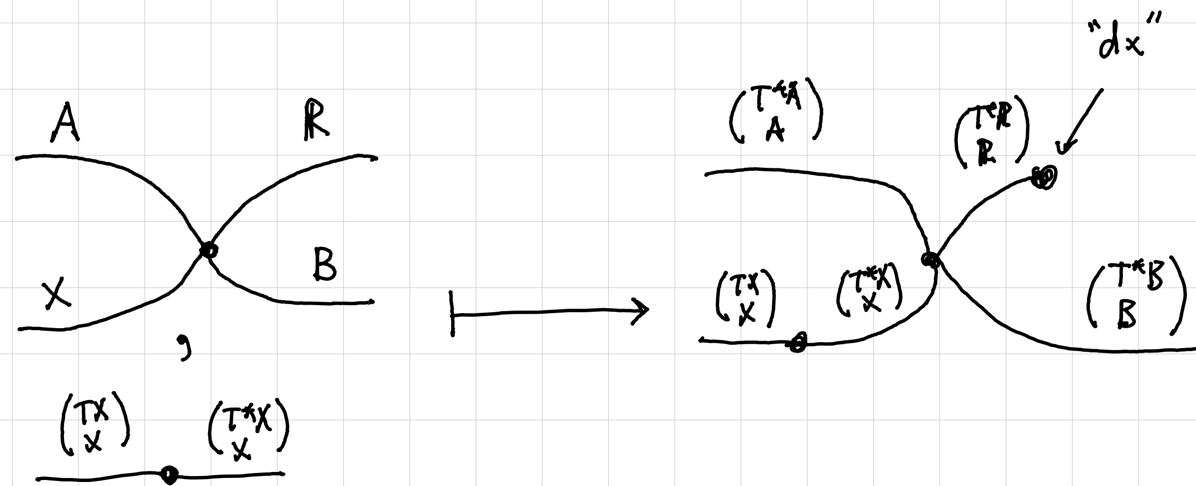
$$\begin{tikzcd}
T^*X_1 \arrow[r] \arrow[dr] & TX_1 \arrow[d] & T^*X_2 \arrow[r] \arrow[dr] & TX_2 \arrow[d] \\
& X_1 & & X_2
\end{tikzcd}$$

$$\mapsto$$

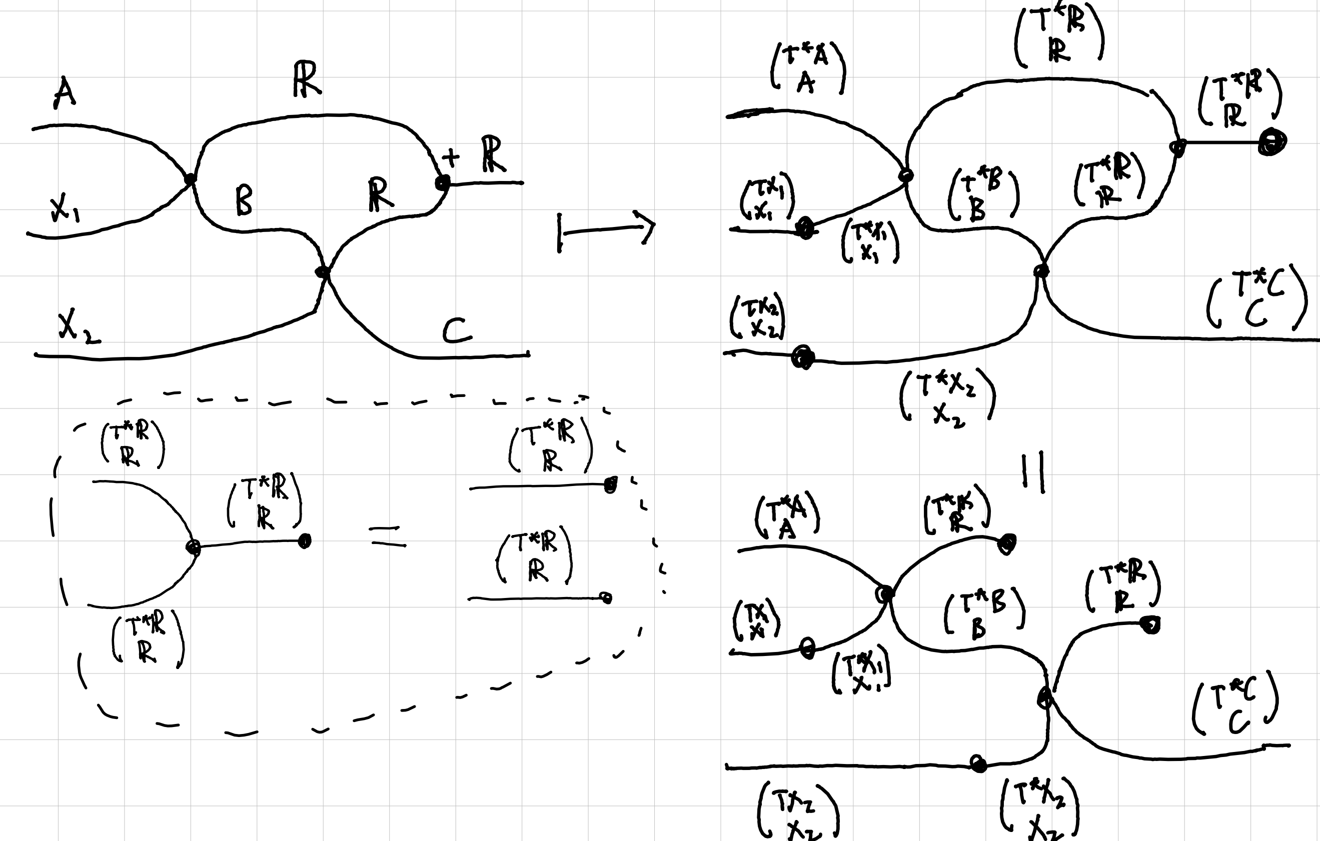
$$\begin{tikzcd}
T^*X_1 \times T^*X_2 \arrow[r] \arrow[dr] & TX_1 \times TX_2 \arrow[d] \\
& X_1 \times X_2
\end{tikzcd}$$



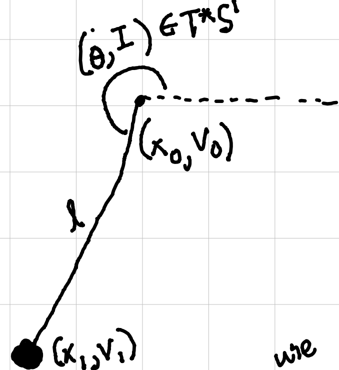
# OpenErg to COrg



# Why is the semantics functorial?



# 9 The open pendulum




$$x_1 = x_0 + l\,\hat{\theta}$$

$$v_1 = v_0 + \frac{I}{L}\,\widehat{\left(\theta + \frac{\pi}{2}\right)}$$

$$H = \underbrace{\frac{1}{2} m \|v_1\|^2}_{\text{kinetic energy}} + \underbrace{m g\, x_{1,1}}_{\text{potential energy}}$$

use standard Poisson structure

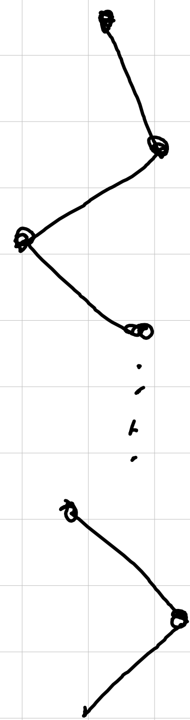
$$\mathbb{R}^4 \times T^*S^1 \xrightarrow{\langle x_1, v_1, H\rangle} \mathbb{R}^4 \times \mathbb{R}$$

$$\Downarrow$$

$$\begin{pmatrix} T^*\mathbb{R}^4 \\ \mathbb{R}^4 \end{pmatrix} \otimes \begin{pmatrix} TT^*S^1 \\ T^*S^1 \end{pmatrix} \rightleftarrows \begin{pmatrix} T^*\mathbb{R}^4 \\ \mathbb{R}^4 \end{pmatrix}$$

## n-fold pendulum

It just works



## Comparison with similar formulations

- Lens-based Double Categorical Systems Theory
    - COrg handles Mealy
    - DCST has representable behaviors (hopefully coming soon for COrg)
- Resource-sharers / resource sharing machines
    - COrg has no relations, only functions
    - COrg is more directed